

ΔV Control Distance with Dynamics Parameter Uncertainty

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Increasing quantities of active spacecraft and debris in the space environment pose substantial risk to the continued safety of specific orbit regimes. Successful quantification of space object state knowledge is a fundamental prerequisite in ensuring operational safety and continuity. However, uncertainty in orbit dynamics such as atmospheric drag, solar radiation pressure, and other effects can confound track association efforts and thereby decrease object state knowledge. Errors in dynamics modeling can cause objects to become lost or incorrectly associated with maneuvers, and need to be accounted for. This paper demonstrates how existing optimal control ΔV object correlation approaches can be modified to account for distributions in dynamics parameters (e.g. atmospheric density, drag coefficients, and surface areas). The method involves linearizing about a nominal connecting trajectory and computing the variation in the required ΔV cost of equivalent maneuvers due to variations in the dynamics parameters and boundary conditions. The resulting variations are then considered to be drawn from distributions, and finally distributions of possible ΔV expenditures are computed. An example of debris in Low Earth Orbit (LEO) is given to demonstrate the utility of the approach and conclusions are made.

I. Introduction & Background

There are currently in excess of 22,000 objects greater than 10cm in Earth orbit.¹ Correlating observations or sequences of observations with individual objects and detecting / characterizing object maneuvers are enabling and persistent endeavors in Space Situational Awareness (SSA).² Due to the disparity between available tracking assets and the increasing quantities of on-orbit objects, correlating observations with objects and detecting / characterizing maneuvers is a challenging pursuit. In particular, unknown maneuvers, uncertain dynamics parameters, boundary condition uncertainty, or long periods without observations can confound traditional orbit determination or association approaches.

Some candidate association metrics involve mapping observations or state knowledge to a common epoch and quantifying the distance between the observations and associated uncertainties (e.g. Euclidean, Mahalanobis,³ Kullback-Liebler,⁴ and Battacharya⁵ distances). Other promising approaches involve mapping observations to feasible volumes using constants of motion and searching for volume intersections.⁶ These approaches make a critical assumption that the object being observed is not actively maneuvering. While this is true in the vast majority of cases, these approaches are not entirely applicable in situations where an object is maneuvering (or has maneuvered).

A different approach developed by the author uses minimum fuel usage as a rigorously defined distance metric with which to associate observations and characterize potential maneuvers.^{7,8} A fundamental assumption in this approach is that operators desire to minimize fuel usage during spacecraft operations. In these papers, two specific problem types, the Uncertain Two Point Boundary Value Problem (UTPBVP), consisting of two full-state boundary conditions, and the Measurement Residual Boundary Value Problem (MRBVP), consisting of a full-state and measurement residual boundary condition, are introduced and addressed. Optimal control theory is used to construct minimum Δv trajectories satisfying the UTPBVP and

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MRBVP boundary conditions, and the respective boundary condition uncertainty distributions mapped to Δv uncertainty distributions. The nominal connecting Δv costs may be used to correlate tracks / observations or to detect maneuvers, given the uncertainty in Δv cost.

This paper addresses an important limitation of the current Δv distance framework. Namely, uncertainty in the dynamics parameters (e.g. drag coefficient, cross-sectional areas, solar radiation pressure) has not been addressed. Unless adequately accounted for, errors in dynamics parameters can manifest as apparent Δv costs, suggesting that a maneuver may have occurred when such is not the case. This paper intends to resolve this shortcoming. The contribution of this paper is the incorporation of dynamics parameter uncertainty distributions in the framework, thus reflecting dynamics uncertainty in the resulting distribution of possible Δv costs. Achieving this goal will reduce the incidence of false-positive maneuver detections when hypothesis testing methods are used.

II. Theory

The distance metric choice and uncertainty approximation from previous research^{7,8} are first summarized. From the Minkowski Inequality and a well known application of the Schwartz Inequality,⁹ the performance index may be rewritten as

$$\begin{aligned} d_{\Delta V} &= \int_{t_0}^{t_f} \|\mathbf{u}_n(\tau) + \delta\mathbf{u}(\tau)\| d\tau \\ &\leq \int_{t_0}^{t_f} \|\mathbf{u}_n(\tau)\| d\tau + \int_{t_0}^{t_f} \|\delta\mathbf{u}(\tau)\| d\tau \\ &\leq \int_{t_0}^{t_f} \|\mathbf{u}_n(\tau)\| d\tau + \sqrt{2(t_f - t_0) \int_{t_0}^{t_f} \frac{1}{2} \delta\mathbf{u}(\tau)^T \delta\mathbf{u}(\tau) d\tau} \end{aligned}$$

Therefore, an alternate performance index that accounts for the nominal minimum fuel trajectory and treats variation as a quadratic cost is written as

$$d_{\Delta V} \leq P_C = \int_{t_0}^{t_f} \|\mathbf{u}_n(\tau)\| d\tau + \sqrt{2(t_f - t_0) \int_{t_0}^{t_f} \frac{1}{2} \delta\mathbf{u}(\tau)^T \delta\mathbf{u}(\tau) d\tau} \quad (1)$$

Given dynamics of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{k}, t) \quad (2)$$

where $\mathbf{x} \in \mathbb{R}^n$ is defined as the state, $\mathbf{u} \in \mathbb{R}^m$ the control input, $\mathbf{k} \in \mathbb{R}^q$ is a set of dynamics parameters, and $t \in [t_0, t_f]$ is time. The adjoint dynamics are defined as

$$\dot{\mathbf{p}} = - \left(\frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{k}, t)}{\partial \mathbf{x}} \right)^T \mathbf{p} \quad (3)$$

with the adjoint states $\mathbf{p} \in \mathbb{R}^n$. Note that by definition $\dot{\mathbf{k}} = \mathbf{0}$. The boundary conditions used here are somewhat more general than either the UTPBVP or MRBVP approaches, and are given as

$$\begin{aligned} \mathbf{g}_0(\mathbf{x}_0, t_0) &= \mathbf{0} \\ \mathbf{g}_f(\mathbf{x}_f, t_f) &= \mathbf{0} \end{aligned} \quad (4)$$

Where $\mathbf{g}_0 \in \mathbb{R}^{p_0}$, $\mathbf{g}_f \in \mathbb{R}^{p_f}$, and it is required only that $p_0 + p_f > n$ (if this is not the case, a homogeneous trajectory can satisfy the boundary conditions). Supposing that a nominal optimal trajectory $(\mathbf{x}(t), \mathbf{p}(t))$ minimizing Eq. (1) subject to dynamics (2) and boundary conditions (4) has been computed, over the interval $t \in [t_0, t_f]$ the state and adjoint may be written as

$$\begin{aligned} \mathbf{x}(t) &= \phi_x(t; \mathbf{x}_0, \mathbf{p}_0, \mathbf{k}, t_0) \\ \mathbf{p}(t) &= \phi_p(t; \mathbf{x}_0, \mathbf{p}_0, \mathbf{k}, t_0) \end{aligned} \quad (5)$$

Taking the Taylor series expansion of the state and adjoint solutions (5) with respect to variations in the initial state \mathbf{x}_0 , adjoint \mathbf{p}_0 , and parameters \mathbf{k} at time $t = t_f$ generates

$$\begin{aligned}\mathbf{x}_f + \delta\mathbf{x}_f &= \phi_x(t_f; \mathbf{x}_0, \mathbf{p}_0, \mathbf{k}, t_0) + \frac{\partial\phi_x}{\partial\mathbf{x}_0}\delta\mathbf{x}_0 + \frac{\partial\phi_x}{\partial\mathbf{p}_0}\delta\mathbf{p}_0 + \frac{\partial\phi_x}{\partial\mathbf{k}}\delta\mathbf{k} + \text{H.O.T.} \\ \mathbf{p}_f + \delta\mathbf{p}_f &= \phi_p(t_f; \mathbf{x}_0, \mathbf{p}_0, \mathbf{k}, t_0) + \frac{\partial\phi_p}{\partial\mathbf{x}_0}\delta\mathbf{x}_0 + \frac{\partial\phi_p}{\partial\mathbf{p}_0}\delta\mathbf{p}_0 + \frac{\partial\phi_p}{\partial\mathbf{k}}\delta\mathbf{k} + \text{H.O.T.}\end{aligned}$$

Applying the identity (5) and keeping only first-order variation terms, the linearized relationship between the boundary condition and parameter variations may be written as

$$\begin{bmatrix} \delta\mathbf{x}_f \\ \delta\mathbf{p}_f \end{bmatrix} = \begin{bmatrix} \Phi_{xx}(t_f, t_0) & \Phi_{xp}(t_f, t_0) & \Phi_{xk}(t_f, t_0) \\ \Phi_{px}(t_f, t_0) & \Phi_{pp}(t_f, t_0) & \Phi_{pk}(t_f, t_0) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{p}_0 \\ \delta\mathbf{k} \end{bmatrix} \quad (6)$$

Supposing that $\delta\mathbf{x}_0$, $\delta\mathbf{x}_f$, and $\delta\mathbf{k}$ are known variations, the adjoint variations $\delta\mathbf{p}_0$ and $\delta\mathbf{p}_f$ may be computed as

$$\delta\mathbf{p}_0 = \Phi_{xp}(t_f, t_0)^{-1} \begin{bmatrix} -\Phi_{xx}(t_f, t_0) & \mathbb{I} & \Phi_{xk}(t_f, t_0) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \\ \delta\mathbf{k} \end{bmatrix} \quad (7)$$

The linear variation in the adjoint $\delta\mathbf{p}(t)$ over the interval $t \in [t_0, t_f]$, may be then written as

$$\delta\mathbf{p}(t) = \Phi_{px}(t, t_0)\delta\mathbf{x}_0 + \Phi_{pp}(t, t_0)\Phi_{xp}(t_f, t_0)^{-1} \begin{bmatrix} -\Phi_{xx}(t_f, t_0) & \mathbb{I} & \Phi_{xk}(t_f, t_0) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \\ \delta\mathbf{k} \end{bmatrix} + \Phi_{pk}(t, t_0)\delta\mathbf{k}$$

which may be simplified to

$$\delta\mathbf{p}(t) = \Lambda(t; t_0, t_f) \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \\ \delta\mathbf{k} \end{bmatrix} = \begin{bmatrix} \Lambda_0(t; t_0, t_f) & \Lambda_f(t; t_0, t_f) & \Lambda_k(t; t_0, t_f) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \\ \delta\mathbf{k} \end{bmatrix} \quad (8)$$

where

$$\begin{aligned}\Lambda_0(t; t_0, t_f) &= \Phi_{px}(t, t_0) - \Phi_{pp}(t, t_0)\Phi_{xp}(t_f, t_0)^{-1}\Phi_{xx}(t_f, t_0) \\ \Lambda_f(t; t_0, t_f) &= \Phi_{pp}(t, t_0)\Phi_{xp}(t_f, t_0)^{-1} \\ \Lambda_k(t; t_0, t_f) &= \Phi_{pk}(t, t_0) + \Phi_{pp}(t, t_0)\Phi_{xp}(t_f, t_0)^{-1}\Phi_{xk}(t_f, t_0)\end{aligned}$$

To compute the state transition matrices, the time derivative of Eq. (6) over the time interval $[t, t_0]$ with fixed variations $\delta\mathbf{x}_0$, $\delta\mathbf{p}_0$, and $\delta\mathbf{k}$ is taken:

$$\frac{d}{dt} \begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{p} \end{bmatrix} = \frac{d}{dt} \begin{bmatrix} \Phi_{xx}(t, t_0) & \Phi_{xp}(t, t_0) & \Phi_{xk}(t, t_0) \\ \Phi_{px}(t, t_0) & \Phi_{pp}(t, t_0) & \Phi_{pk}(t, t_0) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{p}_0 \\ \delta\mathbf{k} \end{bmatrix}$$

For the linearized system about a nominal trajectory,

$$\frac{d}{dt} \begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial\mathbf{f}}{\partial\mathbf{x}} & \frac{\partial\mathbf{f}}{\partial\mathbf{p}} & \frac{\partial\mathbf{f}}{\partial\mathbf{k}} \\ -\frac{\partial}{\partial\mathbf{x}} \left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T \mathbf{p} \right) & -\frac{\partial\mathbf{f}}{\partial\mathbf{x}} & -\frac{\partial}{\partial\mathbf{k}} \left(\frac{\partial\mathbf{f}}{\partial\mathbf{x}}^T \mathbf{p} \right) \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{p} \\ \delta\mathbf{k} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{xx} & \mathbf{A}_{xp} & \mathbf{A}_{xk} \\ \mathbf{A}_{px} & \mathbf{A}_{pp} & \mathbf{A}_{pk} \end{bmatrix} \begin{bmatrix} \delta\mathbf{x} \\ \delta\mathbf{p} \\ \delta\mathbf{k} \end{bmatrix}$$

Using the definition of the state transition matrix (6), for all variations $\delta\mathbf{x}_0$, $\delta\mathbf{p}_0$, and $\delta\mathbf{k}$, the state transition dynamics may be written as

$$\frac{d}{dt} \begin{bmatrix} \Phi_{xx}(t, t_0) & \Phi_{xp}(t, t_0) & \Phi_{xk}(t, t_0) \\ \Phi_{px}(t, t_0) & \Phi_{pp}(t, t_0) & \Phi_{pk}(t, t_0) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{xx} & \mathbf{A}_{xp} & \mathbf{A}_{xk} \\ \mathbf{A}_{px} & \mathbf{A}_{pp} & \mathbf{A}_{pk} \end{bmatrix} \begin{bmatrix} \Phi_{xx}(t, t_0) & \Phi_{xp}(t, t_0) & \Phi_{xk}(t, t_0) \\ \Phi_{px}(t, t_0) & \Phi_{pp}(t, t_0) & \Phi_{pk}(t, t_0) \\ \mathbf{0} & \mathbf{0} & \mathbb{I} \end{bmatrix} \quad (9)$$

with the initial condition

$$\Phi(t_0, t_0) = \begin{bmatrix} \Phi_{xx}(t_0, t_0) & \Phi_{xp}(t_0, t_0) & \Phi_{xk}(t_0, t_0) \\ \Phi_{px}(t_0, t_0) & \Phi_{pp}(t_0, t_0) & \Phi_{pk}(t_0, t_0) \end{bmatrix} = \begin{bmatrix} \mathbb{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} & \mathbf{0} \end{bmatrix}$$

Using Eq. (8), the variation in the optimal control may be computed. Variations in the optimal control policy $\delta\mathbf{u}(t)$ about the nominal optimal primer vector solution $\mathbf{u}^*(t)$ given linearized variations in the initial state $\delta\mathbf{x}_0$, final state $\delta\mathbf{x}_f$, and dynamics parameters $\delta\mathbf{k}$ may be expressed as

$$\delta\mathbf{u}(t) = -\frac{\partial \mathbf{f}^T}{\partial \mathbf{u}} \mathbf{\Lambda}(t; t_0, t_f) \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \\ \delta\mathbf{k} \end{bmatrix} \quad (10)$$

Defining the variational parameter $\delta\mathbf{z}$ as

$$\delta\mathbf{z}^T = \begin{bmatrix} \delta\mathbf{x}_0^T & \delta\mathbf{x}_f^T & \delta\mathbf{k}^T \end{bmatrix} \quad (11)$$

The variational ΔV cost defined by P_C given variations in the initial state $\delta\mathbf{x}_0$, final state $\delta\mathbf{x}_f$, and dynamics parameters $\delta\mathbf{k}$ is

$$P_C = \sqrt{2(t_f - t_0) \int_{t_0}^{t_f} \delta\mathbf{z}^T \mathbf{\Lambda}(\tau; t_0, t_f)^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \frac{\partial \mathbf{f}^T}{\partial \mathbf{u}} \mathbf{\Lambda}(\tau; t_0, t_f) \delta\mathbf{z} d\tau}$$

Because $\delta\mathbf{z}$ does not depend on τ ,

$$P_C = \sqrt{2(t_f - t_0) \delta\mathbf{z}^T \mathbf{\Omega}(t_f, t_0) \delta\mathbf{z}} \quad (12)$$

where

$$\mathbf{\Omega}(t_f, t_0) = \int_{t_0}^{t_f} \mathbf{\Lambda}(\tau; t_0, t_f)^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \frac{\partial \mathbf{f}^T}{\partial \mathbf{u}} \mathbf{\Lambda}(\tau; t_0, t_f) d\tau \quad (13)$$

with $\mathbf{\Omega}(t_0, t_0) = \mathbf{0}$. If the variations $\delta\mathbf{z}$ are considered random variables $\delta\mathbf{Z}$ such that $\delta\mathbf{Z} \sim N(\mathbf{0}, \mathbf{P}_z)$, where

$$\mathbf{P}_z = \begin{bmatrix} \mathbf{P}_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_k \end{bmatrix}$$

then the random quadratic cost associated with the boundary condition and dynamics parameter uncertainty may be written as

$$P_u = \delta\mathbf{Z}^T \mathbf{\Omega}(t_f, t_0) \delta\mathbf{Z} \quad (14)$$

In previous work⁷ the first and second moments of P_u are derived analytically. Because Eq. (14) has the same form as the quadratic cost in the previous work, if $\delta\mathbf{Z}$ is Gaussian, the analytical first and second moment results may be applied here. They are

$$\mathbb{E}[P_u] = \mu_{P_u} = \text{Tr}[\mathbf{\Omega} \mathbf{P}_z] \quad (15)$$

and

$$\mathbb{E} \left[\mathbb{E} [P_u]^2 - P_u^2 \right] = \sigma_{P_u}^2 = 2\text{Tr} \left[(\Omega \mathbf{P}_z)^2 \right] \quad (16)$$

Similarly, the bounding random Δv cost associated with the boundary condition and dynamics parameter uncertainty is written as

$$P_{\Delta v, u} = \sqrt{2(t_f - t_0)P_u} \quad (17)$$

Qualitatively, it can be seen that as $\mathbf{P}_z \rightarrow \mathbf{0}$, the cost distribution collapses to the nominal Δv control distance $d_{\Delta v}$. Additionally, as the dynamics uncertainty defined by \mathbf{Q}_k decreases, the cost uncertainty will match the uncertainty found using the UTPBVP or MRBVP approach derived in previous work.^{7,8} Conversely, as the dynamics parameter uncertainty strictly increases (holding boundary condition uncertainties constant), Δv cost distribution variance and mean strictly increase, as indicated by Eqs. (15) and (16). Ultimately this has the effect of reducing the incidence of false-positive maneuver detection in the presence of dynamics parameter uncertainty using the hypothesis testing methods discussed in Holzinger et al.⁷ Thus, the extension in this paper to existing theory incorporating dynamics parameter uncertainty into the quadratic variational cost about nominal optimal connecting trajectories is successful; the extension reduces the incidence of false positive maneuver detection when hypothesis testing is used.

III. Conclusions

A short summary of related past work is given and the problem of dynamics parameter uncertainty is motivated. Existing approaches to quantifying uncertainty in connecting trajectory cost are extended by carefully accounting for dynamics parameter variations on the optimal cost. The variations in the optimal cost due to dynamics parameter uncertainty are combined with those of boundary condition uncertainty to quantify total quadratic and Δv cost uncertainty. The statistics of the Δv cost uncertainty are discussed, and holding boundary condition uncertainty constant, the effects of increasing or decreasing dynamics parameter uncertainty are shown to increase or decrease the quadratic and Δv cost uncertainties correspondingly.

Future work is focused on extending the current approach to accept arbitrary numbers of general boundary condition constraints, each corresponding to measurements or a-priori information. Also, further efforts will attempt to decrease the conservatism of the current approach with respect to Δv cost distribution conservatism.

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