Object Correlation Using Control Effort Metrics with Boundary Condition Uncertainties

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Analytical formalisms are used to compute control distance metrics between Uncorrelated Tracks (UCTs) with large observation gaps and uncertain boundary conditions. Control usage as a correlation metric is discussed and shown to be appropriate for correlation of on-orbit Resident Space Object (RSO) tracks when the operator is concerned with fuel usage minimization. An optimal control approach is used to compute the nominal connecting control cost as well as changes in cost due to boundary condition variations. The resulting distributions of the control correlation metric are then modeled using Pearson's approximation and an approach to rank correlations is presented. Examples are given including the effects of J_2 and J_3 that demonstrate how the correlation approach may be applied.

I. Introduction

Increasing quantities of active payloads and trackable orbital debris[2,3] provide excellent motivation to investigate analytical methods by which Uncorrelated Tracks (UCTs) of maneuvering Resident Space Objects (RSOs), particularly those within close proximity, may be associated with on another.

In the vast majority of cases object track correlation methods use a concept of 'state distance' to probabilistically associate one track with another (least-squares, minimum variance projections, etc.). Often this measure is a function of the computed difference between a homogeneous propagation of the last known state estimate and the newly observed object state estimate. While perfectly admissible, the use of a 'state distance' measure for UCT correlation is not ideal for objects that conduct maneuvers during gaps in observation. Rather, an intuitive approach to capture the idea of 'distance' for maneuvering spacecraft is to construct a metric that quantifies the control usage necessary to connect a previous track (old UCT) with a newly acquired object (new UCT). This method benefits from that fact that the state distance is necessarily zero when the control effort metric is zero.

This paper demonstrates the utility of an approach[5] that accounts for boundary condition uncertainties while using a control metric performance index to sensibly rank UCTs to one another. It is envisioned that such a framework can serve as an auxiliary tool for existing on-orbit UCT matching methods, as well as provide insight into required control authorities. A rigorous Hamiltonian-based optimal control approach is used to consider variations of trajectories about a nominal optimal connecting orbit. The UCT state uncertainties are incorporated into the problem and the resulting probability density function and cumulative distribution function of control effort metrics are determined. The concept of stochastic dominance is borrowed from mathematical finance and used to sensibly rank candidate UCT pairing control effort metric statistics. A proximity operations example is given where two sets of UCT pairings are evaluated and ranked within this framework. Operational insights, conclusions, and future work are discussed.

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II. Problem Definition

The problem under consideration is illustrated in Figure 1. In this scenario, an initial object track consisting of a sequence of observations ultimately produces a state estimate $\mathbf{x}_0$ and an associated estimate covariance $\mathbf{P}_0$ (corresponding to some arbitrary time t_0). We define this track UCT_0 as the triplet $(t_0, \mathbf{x}_0, \mathbf{P}_0)$. At some later time t_f, a new object track is initiated based on new observations. After all of the new observations are collected an estimate of the state and covariance when the new track was first started (at time t_f) can be generated, creating the new UCT_f triplet $(t_f, \mathbf{x}_f, \mathbf{P}_f)$. The estimation theory used in to generate UCTs can be found in Tapley.[4]

Supposing now that multiple initial and final UCTs exist, we are faced with the problem of determining which UCTs should be associated (or 'paired') with one another. One way to do this is to compute a measure of how 'expensive' a maneuver between UCTs would be. A logical assumption would be that UCTs with the smallest required connecting control effort should be paired to one another, as on-orbit fuel is such a scarce commodity. This concept is very similar to comparing differences in propagated homogeneous states $\mathbf{x}_{f,p}$ to new UCT states $\mathbf{x}_f$, as if $\mathbf{x}_{f,p} \approx \mathbf{x}_f$, the minimum optimal control is necessarily $\mathbf{u}^*(t) = \mathbf{0}$.

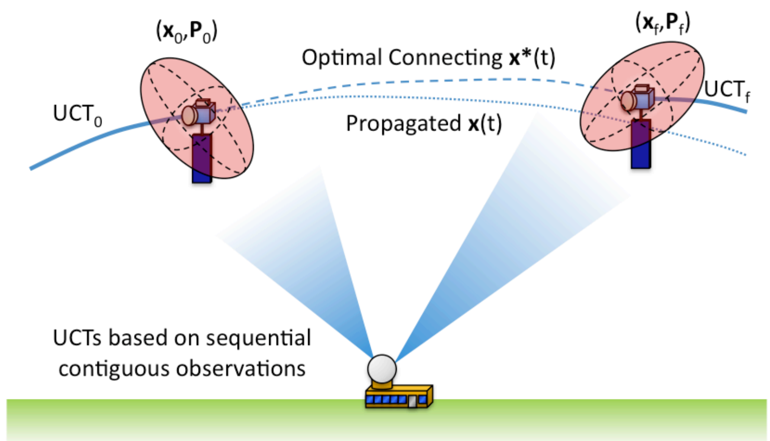


Figure 1. Problem Illustration

Since operational boundary conditions are estimates with uncertainty distributions, the resulting distribution of the performance metric P must be determined. To that end the definitions of the performance function and boundary conditions are now given.

A. Performance Function

One candidate trajectory performance function that approximates fuel cost is

$$P = \frac{1}{2}\int_{t_0}^{t_f} \mathcal{L}_u(\mathbf{x}(\tau), \mathbf{u}(\tau), \tau)d\tau = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{u}(\tau)^T\mathbf{u}(\tau)d\tau \tag{1}$$

Subject to dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$ with $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, and $t \in [t_0, t_f]$. This performance function is similar to an energy measure and closely resembles the analogous *Control Distance* norm

$$P_{L_2} = \sqrt{2P} = \|\mathbf{u}(t)\|_{L_2} = \left(\int_{t_0}^{t_f} \mathbf{u}(\tau)^T\mathbf{u}(\tau)d\tau\right)^{\frac{1}{2}}$$

which is an example of an L_2 norm. By inspection, it is clear that any optimal trajectory $(\mathbf{x}^*, \mathbf{u}^*)$ that minimizes P also minimizes the L_2-norm $P_{L_2} = \sqrt{2P}$. Ultimately, the norm of interest will measure the total change in velocity ΔV:

$$P_{\Delta V} = \Delta V = \int_{t_0}^{t_f} \|\mathbf{u}(\tau)\|_2 d\tau = \int_{t_0}^{t_f} \sqrt{\mathbf{u}(\tau)^T\mathbf{u}(\tau)}d\tau$$

Minimum $P_{\Delta V}$ (fuel) problems are much more difficult to solve than minimum energy analogs such as P. In the interest of progressing the theory summarized in the next section, this effort simply uses P. Using the Cauchy-Schwarz inequality it can be shown[6] that the performance functions P and P_{L_2} bound ΔV from above:

$$\Delta V \leq \sqrt{t_f - t_0} P_{L_2} = \sqrt{t_f - t_0}\sqrt{2P} \tag{2}$$

Thus, the performance function P, an energy cost analog, can produce an upper conservative bound on the possible fuel cost distribution. The function P_{L_2} can be used to define a normed linear space in which to make performance rankings. Because P_{L_2} is a norm in this space, it also satisfies the properties of a metric.

B. Boundary Conditions

Traditionally, connecting two UCTs using control is the classic Two-Point Boundary Value Problem (TPBVP). The modification made to the TPBVP for this analysis is that $\mathbf{x}_0$ and $\mathbf{x}_f$ are not known exactly, and are described as random vectors $\mathbf{X}_0$ and $\mathbf{X}_f$, respectively. We call this problem the Uncertain Two-Point Boundary Value Problem (UTPBVP). The distributions of $\mathbf{X}_0$ and $\mathbf{X}_f$ are assumed to be Guassian, with $\mathbf{X}_0 \in N(\mathbf{x}_0, \mathbf{P}_0)$ and $\mathbf{X}_f \in N(\mathbf{x}_f, \mathbf{P}_f)$. Though this paper uses Guassian distributions for the boundary conditions, the optimal control results summarized in the Theory section are valid for arbitrary distributions about the nominal boundary values. With these definitions in hand, the general nonlinear dynamics case is now examined in detail.

III. Theory

The nominal optimal trajectory problem for a general nonlinear system is now solved using Hamiltonian formalisms.[7–9] After this solution is found, variations in the boundary conditions are considered and their impact on the performance function evaluated. For nonlinear systems with a performance function P the Hamiltonian $\mathcal{H}$ and optimal control $\mathbf{u}^*$ are

$$\mathcal{H} = \inf_{\mathbf{u}} \left[\frac{1}{2}\mathbf{u}^T\mathbf{u} + \mathbf{p}^T\mathbf{f}(\mathbf{x}, \mathbf{u}, t) \right] \tag{3}$$

$$\mathbf{u}^* = -\frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T \mathbf{p} \tag{4}$$

with the second-order necessary condition $[\partial^2\mathcal{H}/\partial\mathbf{u}^2] \geq 0$ satisfied. The state and co-state dynamics are then

$$\frac{\partial \mathcal{H}}{\partial \mathbf{p}} = \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}^*, t) \tag{5}$$

$$-\frac{\partial \mathcal{H}}{\partial \mathbf{x}} = \dot{\mathbf{p}} = -\frac{\partial \mathbf{f}}{\partial \mathbf{x}}^T \mathbf{p} \tag{6}$$

Because these equations are nonlinear an analytical solution may not exist. Now a nominal optimal solution $(\mathbf{x}_n(t), \mathbf{u}_n(t))$ connecting the nominal boundary conditions $\mathbf{x}_0$ and $\mathbf{x}_f$ is assumed (which may be numerical), and is expressed as

$$\mathbf{x}_n(t) = \phi_x(t; \mathbf{x}_0, \mathbf{p}_0, t_0) \tag{7}$$

$$\mathbf{p}_n(t) = \phi_p(t; \mathbf{x}_0, \mathbf{p}_0, t_0) \tag{8}$$

for times up to and including t_f. Previous results[5] are now leveraged that linearize about the nominal trajectory $(\mathbf{x}_n(t), \mathbf{p}_n(t))$ to compute variations in the optimal control $\delta\mathbf{u}(t)$ based on variations in the boundary conditions $\delta\mathbf{x}_0$ and $\delta\mathbf{x}_f$.

$$\delta\mathbf{p}(t) = \begin{bmatrix} \mathbf{\Phi}_{px}(t,t_0) - \mathbf{\Phi}_{pp}(t,t_0)\mathbf{\Phi}_{xp}(t_f,t_0)^\dagger\mathbf{\Phi}_{xx}(t_f,t_0) & \mathbf{\Phi}_{pp}(t,t_0)\mathbf{\Phi}_{xp}(t_f,t_0)^\dagger \end{bmatrix} \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix} = \mathbf{\Lambda}(t,t_0) \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix} \tag{9}$$

Note that some of the state transition matrix portions are computed over the interval $[t_0, t]$, while others are computed over $[t_0, t_f]$. After this computation, there now exists a function $\mathbf{\Lambda}(t, t_0)$ that maps variations in the initial and final states to variations in the co-state $\delta\mathbf{p}(t)$ at time t. Since the optimal control is defined in terms of the co-state, linear variations in the control are written as

$$\mathbf{u}^*(t) = \mathbf{u}_n(t) + \delta\mathbf{u}(t) \approx \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T (\mathbf{p}_n(t) + \delta\mathbf{p}(t))$$

Because $\mathbf{u}_n(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T \mathbf{p}_n(t)$, the linear variation in the control is:

$$\delta\mathbf{u}(t) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T \mathbf{\Lambda}(t, t_0) \begin{bmatrix} \delta\mathbf{x}_0 \\ \delta\mathbf{x}_f \end{bmatrix} \tag{10}$$

Recall that $\partial\mathbf{f}/\partial\mathbf{u}$ is evaluated along the nominal optimal trajectory $(\mathbf{x}_n(t), \mathbf{u}_n(t))$. Returning now to the performance function P defined in (1),

$$P = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{u}^*(\tau)^T \mathbf{u}^*(\tau) d\tau,$$

and substituting $\mathbf{u}^*(\tau) = \mathbf{u}_n(\tau) + \delta\mathbf{u}(\tau)$, the performance function becomes

$$P = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{u}_n(\tau)^T \mathbf{u}_n(\tau) d\tau + \int_{t_0}^{t_f} \mathbf{u}_n(\tau)^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T \mathbf{\Lambda}(\tau, 0)\delta\mathbf{z} d\tau + \frac{1}{2}\int_{t_0}^{t_f} \delta\mathbf{z}^T \mathbf{\Lambda}(\tau, 0)^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T \mathbf{\Lambda}(\tau, 0)\delta\mathbf{z} d\tau$$

where $\delta\mathbf{z}^T = \left[\delta\mathbf{x}_0^T \ \delta\mathbf{x}_f^T\right] \in \mathbb{R}^{2n}$. The variable $\delta\mathbf{z}$ does not depend on τ, so the following definitions are made:

$$P_n = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{u}_n(\tau)^T \mathbf{u}_n(\tau) d\tau, \tag{11}$$

$$\omega(t_f, t_0) = \int_{t_0}^{t_f} \mathbf{\Lambda}(\tau, 0)^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \mathbf{u}_n(\tau) d\tau, \tag{12}$$

and

$$\mathbf{\Omega}(t_f, t_0) = \frac{1}{2}\int_{t_0}^{t_f} \mathbf{\Lambda}(\tau, 0)^T \frac{\partial \mathbf{f}}{\partial \mathbf{u}} \frac{\partial \mathbf{f}}{\partial \mathbf{u}}^T \mathbf{\Lambda}(\tau, 0) d\tau, \tag{13}$$

Note that P_n is the performance of the nominal optimal trajectory $(\mathbf{x}_n(t), \mathbf{p}_n(t))$. The performance index P is

$$P = P_n + \omega(t_f, t_0)^T \delta\mathbf{z} + \delta\mathbf{z}^T \mathbf{\Omega}(t_f, t_0)\delta\mathbf{z}$$

The approximation of P in the linear space about $(\mathbf{x}_n(t), \mathbf{p}_n(t))$ is a quadratic form in terms of the boundary condition variations, $\delta\mathbf{z}$. Assumptions must now be made about the distribution of the boundary conditions. From §II.B, $\delta\mathbf{z}$ may be treated as the realization of a Guassian random vector $\delta\mathbf{Z} \in N(\mathbf{0}, \mathbf{P}_z)$, where

$$\mathbb{E}\left[\delta\mathbf{Z}\delta\mathbf{Z}^T\right] = \mathbf{P}_z = \begin{bmatrix} \mathbf{P}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_f \end{bmatrix} \tag{14}$$

Rewriting the performance function as a scalar random variable the following form is obtained:

$$P = P_n + \omega(t_f, t_0)^T \delta\mathbf{Z} + \delta\mathbf{Z}^T \mathbf{\Omega}(t_f, t_0)\delta\mathbf{Z} \tag{15}$$

The quadratic form shown in (15) is an intuitive result, as if the variations in the boundary conditions are reduced to zero, $P = P_n$. Similarly, if the variance in the boundary conditions is large, one would expect the possible values of P to increase. Now, the question of computing the PDF of P arises. Because (15) has a quadratic form, existing theory involving quadratic forms of normal random variables may be applied.

To apply existing results P must be transformed into a standard non-central quadratic form. To do so the following definitions are made:

$$\underline{P}_n = P_n - \frac{1}{4}\omega^T \mathbf{\Omega}^{\dagger}\omega \tag{16}$$

$$\mu_X = \frac{1}{2}\mathbf{\Omega}^{\dagger}\omega \tag{17}$$

$$\mathbf{B}\mathbf{B}^T = \mathbf{P}_z \tag{18}$$

$$\mathbf{b} = \mathbf{T}^T\mathbf{B}^T\mu_X \tag{19}$$

$$\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{2n} \end{bmatrix} = \mathbf{T}^T\mathbf{B}^T\Omega(t_f, t_0)\mathbf{B}\mathbf{T} \tag{20}$$

The matrix $\mathbf{B}$ is a matrix square root of $\mathbf{P}_z$ (or any decomposition such that $\mathbf{B}\mathbf{B}^T = \mathbf{P}_z$). Also, the matrix $\mathbf{T} \in \mathbb{R}^{2n\times 2n}$ is an orthonormal transformation matrix such that $\mathbf{T}^T\mathbf{B}^T\Omega(t_f, t_0)\mathbf{B}\mathbf{T}$ is diagonalized. These definitions transform P to

$$P = \underline{P}_n + \sum_{i=1}^{2n} \lambda_i \left(U_i + b_i\right)^2 \tag{21}$$

Pearson's Approximation[11] is used to capture the first three moments of the true distribution of P. The approximation is

$$P \approx \frac{\theta_3}{\theta_2}\chi_v^2 - \frac{\theta_2^2}{\theta_3} + \theta_1 + \underline{P}_n \tag{22}$$

where

$$\theta_s = \sum_{j=1}^{2n} \lambda_j^s (1 + s b_j^2), \;\; s = 1, 2, 3$$

and the degree of freedom v is

$$v = \frac{\theta_2^3}{\theta_3^2}$$

A. Choosing Sensible UCT Pairings

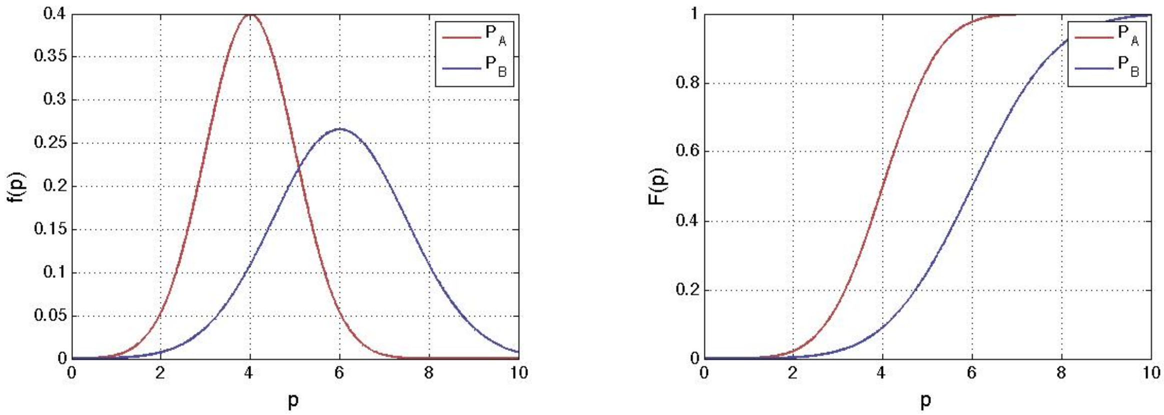
The purpose behind choosing a performance function P that is similar to an L_2-norm was to determine the distribution of analog *Control Distances* of a candidate optimal trajectory between UCTs in a normed linear space. This particular method is used because on-orbit control is quite expensive; it is logical to assume that a minimum of control is used over the interval $[t_0, t_f]$. The ability to generate a probability distribution in the control distance based on uncertainty in the UCTs allows a probabilistic approach when determining the likelihood that two specific UCTs represent the same object.

Unfortunately, this probabilistic approach also means that often there will be cases where two (or more) potential UCT pairings have overlapping control distance distributions. Figure 2(a) illustrates such a situation in which there is a nontrivial overlap between two different hypothetical UCT pairings.

The question now faced is how to rank UCT pairings sensibly. Figure 2(b) illustrates how a UCT pairing may rationally be made. Ideally, for every performance function value of $0 < p < \infty$, $F_A(p) > F_B(p)$. This is equivalent to choosing an arbitrary p and guaranteeing that $\mathbb{P}[P_A \leq p] \geq \mathbb{P}[P_B \leq p]$. In words, this means that it is more likely that UCT pairing A has a smaller *Control Distance* than UCT pairing B *for all possible performance function values in a normed linear space*. This is called stochastic dominance.[12]

Definition III.1. ***Order-q Dominance:***
The distribution $f_A(p)$ is said to ***order-q dominate*** *the distribution $f_B(p)$ if, for all $p \in [0, \infty]$, the following inequality holds:*

$$\mathcal{I}^q\left[f_A(p)\right] \geq \mathcal{I}^q\left[f_B(p)\right] \tag{23}$$



(a) Example of overlapping Control Distance PDFs. Note that there exists a finite and non-trivial probability that either UCT pairing may be correct.

(b) CDF of overlapping Control Distances. Pairing A is clearly more sensible than pairing B

Figure 2. Illustrative PDF and CDF of two candidate UCT Pairings

where $\mathcal{I}[\cdot]$ is the integration operator over $p \in [0, \infty]$. If $q = 1$ ($F_A(p) \geq F_B(p)$), distribution $f_A(p)$ is said to weakly dominate $f_B(p)$.

Brief inspection of (23) shows us that order q dominance implies order $q+1$ dominance. Starting with $q = 0$, this gives the following result

$$\text{0-dom} \Rightarrow \text{1-dom} \Rightarrow \cdots \Rightarrow \text{q-dom}$$

Since 0-dom does not typically occur, the first order dominance that can reasonably be expected is $q = 1$. Order $q = 1$ dominance is equivalent to the cumulative distribution function (cdf) of the distribution of N_a being strictly less that the cdf of the random variable N_b. Orders higher that $q = 1$ do not always have a clear, intuitive meaning. In general, there is no guarantee that there exists an order q such that any two distributions may be ranked.

B. Approximating P for Very Small Boundary Condition Variations

If the boundary condition variations are 'sufficiently small' (such that the term $\delta\mathbf{Z}^T\mathbf{\Omega}(t_f, t_0)\delta\mathbf{Z}$ is negligible), then the approximate performance function P is approximately

$$P = P_n + \omega(t_f, t_0)^T\delta\mathbf{Z} \tag{24}$$

Since $\mathbb{E}[\delta\mathbf{Z}] = \mathbf{0}$,

$$\mathbb{E}[P] = P_n \tag{25}$$

Similarly, the variance of the performance function reduces to

$$\sigma_P^2 = \mathbb{E}\left[\left(P_n + \omega^T\delta\mathbf{Z}\right)\left(P_n + \omega^T\delta\mathbf{Z}\right)^T\right] = \omega(t_f, t_0)^T\mathbf{P}_z\omega(t_f, t_0) \tag{26}$$

More simply, P may be considered a scalar Guassian random variable $P \in N(P_n, \omega(t_f, t_0)^T\mathbf{P}_z\omega(t_f, t_0))$. Needless to say, this simplification greatly eases computational burdens, as once P_n and σ_P^2 have been computed, the PDF is an analytic function:[11]

$$f_{P,approx}(p) = \frac{1}{\sqrt{2\pi\sigma_P^2}}\exp\left(-\frac{(p - P_n)^2}{2\sigma_P^2}\right) \tag{27}$$

As with any assumption, deciding whether $\delta\mathbf{Z}^T\mathbf{\Omega}(t_f, t_0)\delta\mathbf{Z}$ is negligible depends very much on the situation.

IV. Simulation and Results

To demonstrate the applicability of using control distance distributions to correlate or characterize objects two examples are given. The dynamics are Keplerian with J_2 and J_3 perturbations. As indicated in the Theory section, for each initial state / final state combination, a Newton-method-based descent algorithm is used to compute the initial and final co-states that generate the optimal solution to the nominal two point boundary value problem. The variations about the boundary conditions are then modeled as Guassian distributions and the corresponding Pearson's approximation is generated.

A. Example 1: Single Object Demoonstration

This case involves only one object and is meant to demonstrate the application of the approach presented. The boundary conditions and associated uncertainties are shown in Tables 1 and 2. The scenario for

Table 1. Example 1 boundary conditions

	a (km)	e ()	i (deg)	Ω (deg)	ω (deg)	f (deg)
$œ_0$	6770	0.02	-30	0	0	0
$œ_f$	6820	0.03	-30	0	0	180

Table 2. Example 1 boundary condition uncertainty

	σ_x (km)	σ_y (km)	σ_z (km)	$\sigma_{\dot{x}}$ (km/s)	$\sigma_{\dot{y}}$ (km/s)	$\sigma_{\dot{z}}$ (km/s)
UCT_0	0.10	2.00	2.00	0.01	0.03	0.03
UCT_f	0.10	2.00	2.00	0.01	0.03	0.03

Example 1 is illustrated in Figure 3 in the Earth-Centered Inertial (ECI) frame. Because the transfer is over approximately half an orbit period and the eccentricities of the orbits are small, the nominal optimal connecting trajectory is very similar to a traditional Hohman transfer. After computing the nominal optimal connecting trajectory $(\mathbf{x}_n, \mathbf{p}_n)$ both Pearson's approximation and the Guassian approximation of the control distance distribution of P are generated. The distributions are shown in Figure 4. In this case the boundary condition uncertainty is not small so the Guassian approximation is not a good choice.

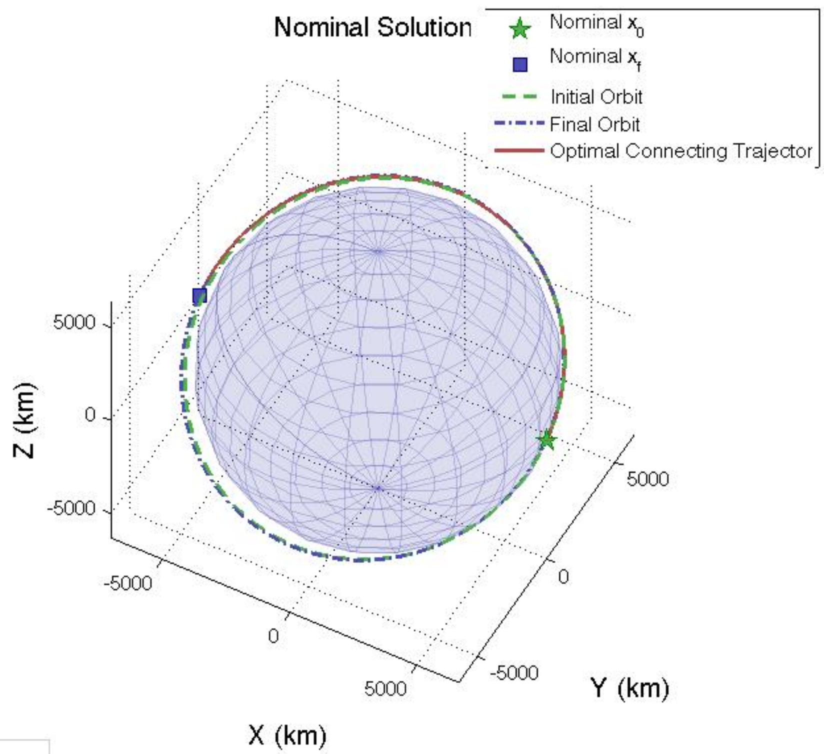


Figure 3. Example 1 ECI visualization

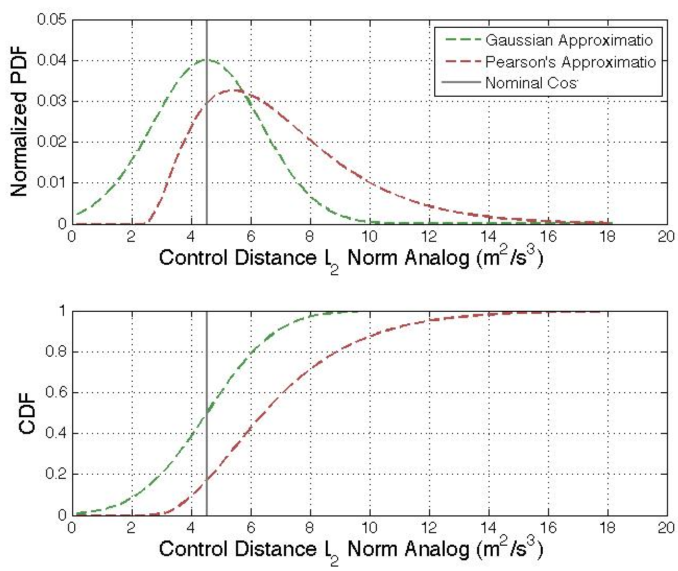


Figure 4. Example 1 control distance distribution for both Pearson's approximation and the Guassian approximation

B. Example 2: Multiple Object Correlation using Control Distance Distributions

This example demonstrates the use of control distance distributions to correlate objects in the presence of maneuvers. In this scenario (see Figure 5), unknown to the observer, the object tracked in $\text{UCT}_{0,1}$ executes a 10 m/s ΔV in the along-track direction immediately after the observation ends. After the observation gap (1/2 of an orbit), the object is not where we expect it, and there is some ambiguity as to which objects can be correlated together. The reference orbit in classical orbit elements $(a, e, i, \Omega, \omega, f)$ for the rotating Hill frame is

$$œ_{ref} = \left[\begin{array}{c} 6770 \text{ km}, 0, 0 \deg, 0 \deg, 0 \deg, 0 \deg \end{array} \right]$$

Figure 5 shows the initial and final nominal states as well as the optimal connecting trajectories between each. The boundary conditions and associated state uncertainties (expressed in relative coordinates of the

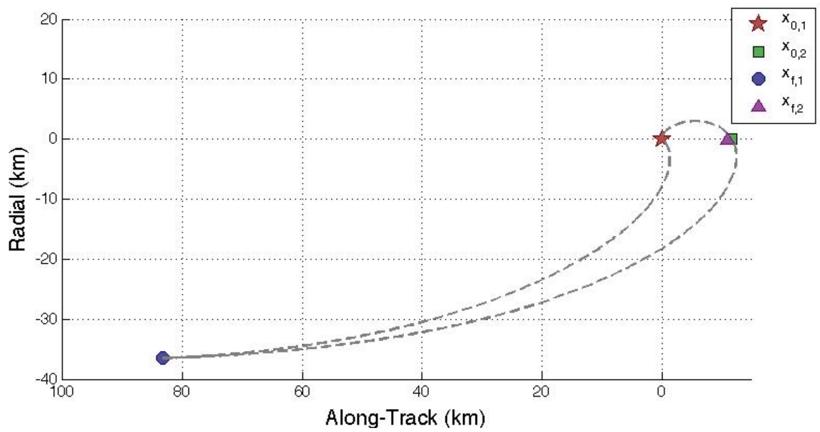


Figure 5. Example 2 Hill-frame visualization of initial and final nominal boundary conditions as well as nominal connecting trajectories

Hill frame) are shown in Tables 3 and 4. The chi-squared approximations of the control distributions for

Table 3. Example 2 boundary conditions in a rotating Hill frame

	r (km)	s (km)	w (km)	$\dot{r}$ (km)	$\dot{s}$ (km)	$\dot{w}$ (km)
$\text{UCT}_{0,1}$	0	0	0	0	0	0
$\text{UCT}_{0,2}$	-0.01	-12	0	0.00027	0	0
$\text{UCT}_{f,1}$	-36	83	0	0.00075	0.07	0
$\text{UCT}_{f,2}$	-0.06	-11	0	-0.00017	0.00011	0

Table 4. Example 2 boundary condition uncertainties

	$\text{UCT}_{0,1}$	$\text{UCT}_{0,2}$	$\text{UCT}_{f,1}$	$\text{UCT}_{f,2}$
σ_r (m)	1	2	2	2
σ_s (m)	10	30	30	40
σ_w (m)	10	30	25	40
$\sigma_{\dot{r}}$ (m/s)	1	1	1	1
$\sigma_{\dot{s}}$ (m/s)	2	4	3	2
$\sigma_{\dot{w}}$ (m/s)	2	4	3	2

each possible UCT pairing mapped into ΔV costs are shown in Figure 6. Upon inspection, it is clear that even 1^{st} order stochastic dominance is not guaranteed between all combinations. To alleviate this ambiguity, we realize that there are several mutually exclusive combinations ($UCT_{0,1}$ cannot map to both $UCT_{f,1}$ and $UCT_{f,2}$). We, therefore, define two cases shown below in Table 5. The combined control distance distributions for each case are shown in Figure 7. Inspecting Figure 7 we see that Case 1 achieves 1^{st} order

Table 5. Example 2 mutually exclusive case descriptions

Case	Description
Case 1	$UCT_{0,1} \rightarrow UCT_{f,1}$ and $UCT_{0,2} \rightarrow UCT_{f,2}$
Case 2	$UCT_{0,1} \rightarrow UCT_{f,2}$ and $UCT_{0,2} \rightarrow UCT_{f,1}$

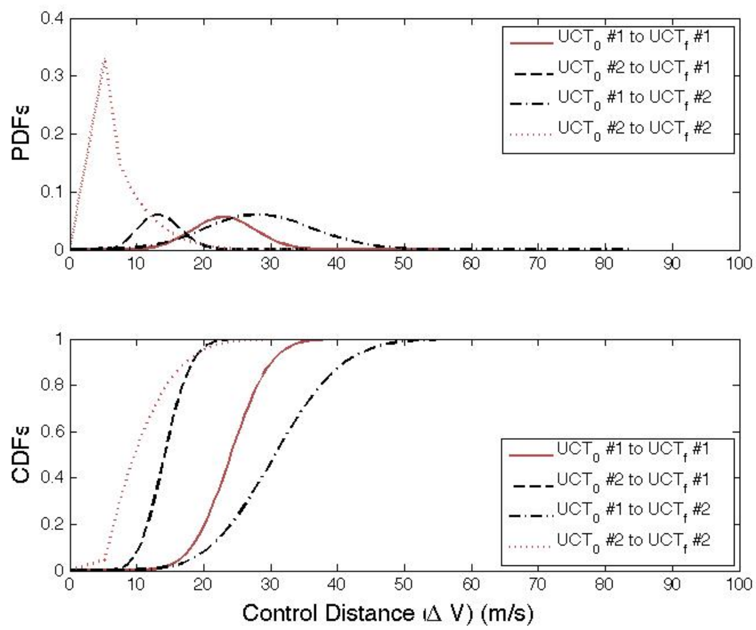


Figure 6. Example 2 control distance distribution for each possible object pairing

dominance over Case 2, so from a fuel usage perspective (control distance), Case 1 is the more likely set of object correlations. Additionally, the control distance distribution may be considered an over estimate of the actual ΔV used in the un-observed maneuvers, and is, therefore, useful for object characterization.

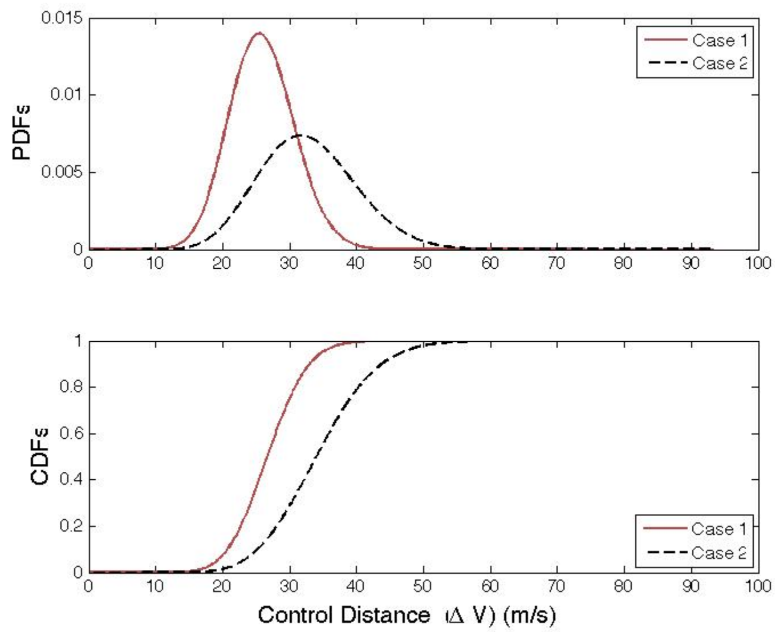


Figure 7. Example 2 case comparison for each mutually exclusive object correlation case

V. Conclusions and Future Work

A framework in which candidate UCT pairings may be ranked in a normed linear space given uncertainties in the UCT boundary conditions is presented. A cost function analogous to an L_2 norm in the control is used, providing many of the advantages and intuitive meanings of traditional distance norms. The boundary conditions were assumed to be Guassian random vectors with arbitrary means and covariance matrices, closely mirroring the data products available in on-orbit operations. The methodology is used for arbitrary nonlinear systems using a rigorous Hamiltonian optimal control approach. The linearization of the space about the nominal optimal connecting trajectory limits this approach to scenarios in which the boundary condition variations are small enough to be sufficiently modeled by linearized dynamics. Variations in the boundary conditions are found to produce a scalar random variable that is a function of a quadratic form in a Guassian random vector. Existing approaches to modeling the PDF and CDF of quadratic forms are leveraged to approximate the distribution as a chi-squared random variable. The concept of stochastic dominance is borrowed from mathematical finance to sensibly rank candidate UCT pairings. In cases where the quadratic term of the boundary condition variations is small, the distribution approaches that of a scalar Guassian distribution and may be analytically determined.

Two examples are given with nonlinear dynamics (Kelplerian, as well as J_2 and J_3) that demonstrate the utility of this approach for object correlation and characterization. The correct UCTs are correlated and an upper bound of the distribution of the ΔV used is computed. This framework has a potential immediate application in SSA operations, as it accounts for the observers own observation uncertainties in providing optimal UCT pairing performance function distributions. The proposed approach computates the nominal optimal control as well as its distribution over time, allowing further investigation into RSO propulsion system performance estimation.

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References

[1] D. P. Lewandowski, "Space Intelligence: Imperative for Space Situational Awareness," AIAA Space 2009 Conference, September, 2009.

[2] O. Brown, P. Eremenko, "Fractionated Space Architectures: A Vision for Responsive Space," AIAA 4^{th} Responsive Space Conference, Los Angeles, 2006.

[3] C. M. Cox, E. J. Degraaf, R. J. Wood, T. H. Crocker, "Intelligent Data Fusion for Improved Space Situational Awareness," AIAA Space 2005 Conference, September 2005.

[4] B. D. Tapley, B. E. Schutz, G. H. Born, "Statistical Orbit Determination," Elsevier Academic Press, Inc, Amsterdam, 2004.

[5] M. J. Holzinger, D. J. Scheeres, "On-Orbit Object Correlation using Control Effort Metrics with Uncertain Boundary Conditions and Measurements," AIAA Conference on Guidance, Navigation, and Control, Toronto, 2010.

[6] E. D. Gustafson, D. J. Scheeres, "Optimal Timing of Control-Law Updates for Unstable Systems with Continuous Control," Journal of Guidance, Control, and Dynamics, Vol. 32, No. 3, MayJune, 2009.

[7] D. F. Lawden, "Analytical Methods of Optimization," Dover Publications Inc., Mineola, NY, 2003.

[8] R. F. Stengel, "Optimal Control and Estimation," Dover Publications, Mineola, 1994.

[9] D. F. Lawden, "Optimal Trajectories for Space Navigation," Butterworths Mathematical Texts, Butterworths, London, 1963.

[10] A. W. Naylor, G. R. Sell, "Linear Operator Theory in Engineering and Science," Applied Mathematical Sciences 40, Springer-Verlag, New York, NY, 2000.

[11] A. M. Mathai, S. B. Provost, "Quadratic Forms in Random Variables: Theory and Applications," Marcel Dekker, Inc., New York, NY, 1992.

[12] A. Meucci, "Risk and Asset Allocation," 3rd Printing, Springer-Verlag, New York, NY, 2007.