THE CONCEPT of an exact solution in modern celestial mechanics

V.A.Antonov

Central Astronomical Observatory at Pulkovo,
196140, St. Petersburg, Pulkovskoe chaussee, 65/1

The concept of an exact solution in modern celestial mechanics and in similar domains of knowledge is bifurcated. On the one hand, we deal with a sufficiently suitable program for numerical calculations. On the other hand, the exact analytical expressions for trajectories in model force fields don't lose their urgency even in the modern epoch. The advantage of an analytical solution consists in the quickness of the study of their properties: as a rule, the relevant formulations can be written in a visible way on little sheets of paper without the repeated use of a computer. On the contrary, the exact analytical solutions do not always appear in the most desirable situations. It is true that nature played us once – in the case of the classical two body problem which is solved in terms of elementary functions. However, very often we deal with the situation where the elegant mathematical solution is related to an artificial model from the point of view of physics and, on the contrary, the naturally formulated and actual problem "does not yield". For instance, it is difficult to find, without a stretch, an application of the problem of the motion around two fixed centres, whose exact solvability has been established by Euler [1]. Nevertheless, some developed methods can be, up to certain degree, transferred from one to the other problem: the study of abstract models assists in better understanding the real motion.

The search for the exact form of trajectories in various fields is developed in different directions that will be enumerated below, even if in an incomplete way.

1) The search, on the off-chance, for a certain mathematical form for trajectories, for example, those satisfying an algebraic equation in x,y,z.

The field with the Hénon-Heiles potential

$$U = \frac{x^2 + \lambda Y^2}{2} + \alpha x^3 + xy^2 \quad (\alpha, \lambda = \text{const})$$

gives us a characteristic example.

We have found [2] all possible trajectories which are expressed by straight lines or second power curves. There are two pairs of oblique rectilinear trajectories and a hyperbolic one at $\alpha < 2/3$; there is an elliptical orbit at $\alpha > 2/3$, and a parabola at

$\alpha = 2/3$, without mentioning the well known motion along the axis $y = 0$ and the special case $\alpha = 2$ with the family of parabolas.

Some of trajectories have been investigated with the aim of testing their stability [3]. The separate algebraic trajectories of a higher degree are known [4], but they are investigated partially.

2) The use of the expansion in a series according to small parameters with the proof of their convergence.

The convergence of direct expansions in powers of small parameters is justified without difficulties [5]. However, it is well known that in this case the structure of the trajectories is not represented correctly, due to the appearance of secular terms. Various averaging methods, according to Delaunay, Lindstedt, Zeipel, etc., eliminate the secular terms, but they do not ensure the complete convergence. As a rule, we know only that the corresponding series are asymptotic. KAM-theory is of great importance, but it has a small domain of practical efficiency.

Perhaps, the estimates following from the KAM-theory are too rough, but there is another aspect of the problem as well.

I suppose that the trajectories under consideration do not objectively belong to the class of the almost periodic ones in the sense of Bohr; therefore, they can't be expressed in terms of trigonometrical functions. We observe this clearly in such special situations as the chaotic motion of some comets, or trajectories of space vehicles approaching planets one after another [6].

3) Variational algorithms and their use in symbolic dynamics.

I have already given a general method for variational determination of a periodic trajectory, with the prescribed rotation number as applied to the problem of the plane mapping of a ring [7], but the propagation to include dynamical problems with continuous time must be almost automatic. In the limit, my construction gives conditional-periodic trajectories [8] in the two-dimensional conservative case. However, there is no guarantee that the trajectory fills an uninterrupted tube in the phase space; slits are possible. Some authors have called such formations as “cantori” [9]. Their construction requires no small parameter.

Apparently, variational algorithms for proving the existence and for constructing the periodic, or conditional-periodic, trajectories are of use, when assigning a more complicated “many-storied” structure. It should be thought that, in the limit case, this complication leads to the outwardly chaotic trajectories. The analogous situation occurs in similar phenomena [10].

4) The complete systems of integrals of motion.

In particular, there is a linear, according to velocities, integral (the angular momentum), or the Jacobi-Stäckel quadratic integral, in the two-dimensional problem with the potential $U(x,y)$. In both cases a sufficient community remains valid; the variety of admissible $U(x,y)$ is equivalent to that of functions of the only argument. As is known, additional integrals are sufficient foundations for the analytical determination of trajectories.

On the contrary, if the integral has a higher degree with respect to velocities, the variety of functions $U(x,y)$ is reduced to that for separate parameters. A kind of isolated small islands in the set of potentials is obtained. For instance, the Bozis integral of the fourth degree can belong to the potential

$$U = \alpha\,[\wp(x) + \wp(y)] + \beta\,[\wp(x+y) + \wp(x-y)] \quad (\alpha, \beta = \text{const})$$

($\wp$ is the Weierstrass elliptical function).

By the way, all the manuals on analytical mechanics state that the existence of an analytical integral allows, in principle, to find all the trajectories. Nevertheless, in the given example a concrete realization of such a program encounters technical difficulties. I know only some special trajectories in this field.

5) Involving the local (particular) invariants of the motion.

Sometimes, we can detect separate invariant surfaces in the phase space that are different from the level surfaces of the energy. Examples are given in [11,12]. Such surfaces can be barriers, hindering a body to turn into another domain of the physical space, or at infinity independent of the degree of complexity of the trajectory itself.

The local invariant permits one to find the exact form of the trajectories in peculiar cases [13], however, the systems under consideration have to rotate as a whole.

There are examples of constructing the invariant manifold, by means of the representation of two quadratic relations with respect to velocity components which do not reduce to the energy integral.

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